REPRODUCTION PURPOSES

REPORT DOCUMENTATION PAGE

Form Approved
OMB NO. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comment regarding this burden estimates or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY *(Leave blank)*

2. REPORT DATE
1996

3. REPORT TYPE AND DATES COVERED
Technical

4. TITLE AND SUBTITLE
Options Pricing in Incomplete Markets: An Asymptotic Approach

5. FUNDING NUMBERS
DAAH04-95-1-0105

6. AUTHOR(S)
Per Aslak Mykland

7. PERFORMING ORGANIZATION NAMES(S) AND ADDRESS(ES)
University of Chicago
Chicago, Illinois 60637

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
U.S. Army Research Office
P.O. Box 12211
Research Triangle Park, NC 27709-2211

10. SPONSORING / MONITORING AGENCY REPORT NUMBER
ARO 31383.5-MA

11. SUPPLEMENTARY NOTES
The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

12a. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release; distribution unlimited.

12 b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 200 words)*

It is explored how incomplete markets can be studied with the help of asymptotics. A compound Poisson model for the stock price is assumed, and an expansion for the price of a European option is obtained as the stock price process converges to a geometric Brownian motion. This formulation also permits one to confront statistical uncertainty in the volatility of the stock price, and we show how this uncertainty impacts on the value of the option.

19961023 228

14. SUBJECT TERMS

15. NUMBER IF PAGES

16. PRICE CODE

17. SECURITY CLASSIFICATION OR REPORT
UNCLASSIFIED

18. SECURITY CLASSIFICATION OF THIS PAGE
UNCLASSIFIED

19. SECURITY CLASSIFICATION OF ABSTRACT
UNCLASSIFIED

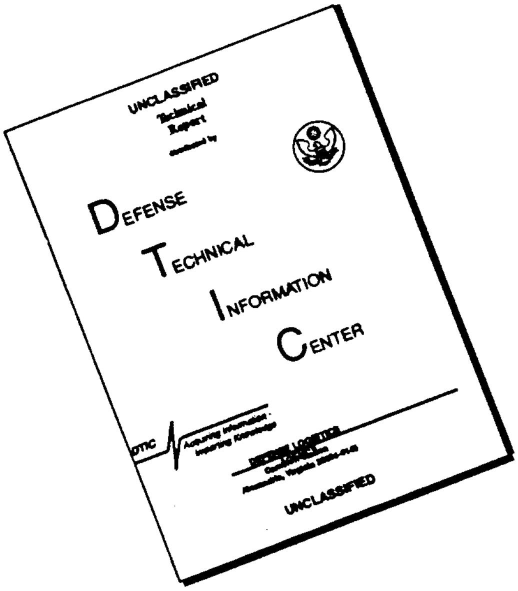
20. LIMITATION OF ABSTRACT
UL

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)
Prescribed by ANSI Std. 239-18
298-102

DTIC QUALITY INSPECTED 1

DISCLAIMER NOTICE



UNCLASSIFIED
Technical
Report
distributed by

DEFENSE
TECHNICAL
INFORMATION
CENTER

DTIC
Acquiring Information -
Imparting Knowledge

UNCLASSIFIED

THIS DOCUMENT IS BEST QUALITY AVAILABLE. THE COPY FURNISHED TO DTIC CONTAINED A SIGNIFICANT NUMBER OF PAGES WHICH DO NOT REPRODUCE LEGIBLY.

Options Pricing in Incomplete Markets: An Asymptotic Approach [1]

by

Per Aslak Mykland

TECHNICAL REPORT NO. 430

Department of Statistics
The University of Chicago
Chicago, Illinois 60637

July 1996

---

[1] The author would like to thank George Constantinides, Xiao-Li Meng, Jostein Paulsen. and Wing Wong for enlightening discussions, and Mitzi Nakatsuka for typing the manuscript.

This research was supported in part by National Science Foundation grants DMS 93-05601 and DMS 96-26266 and Army Research Office grant DAAH04-95-1-0105.

This manuscript was prepared using computer facilities supported in part by the National Science Foundation grants DMS 89-05292 and DMS 87-03942 awarded to the Department of Statistics at The University of Chicago, and by The University of Chicago Block Fund.

Some key words and phrases: Asymptotic Expansion, Incompleteness, Statistical uncertainty.

Options Pricing in Incomplete Markets: An Asymptotic Approach [1]

by

Per Aslak Mykland
The University of Chicago

ABSTRACT

It is explored how incomplete markets can be studied with the help of asymptotics. A compound Poisson model for the stock price is assumed, and an expansion for the price of a European option is obtained as the stock price process converges to a geometric Brownian motion. This formulation also permits one to confront statistical uncertainty in the volatility of the stock price, and we show how this uncertainty impacts on the value of the option.

---

[1] The author would like to thank George Constantinides, Xiao-Li Meng, Jostein Paulsen, and Wing Wong for enlightening discussions. and Mitzi Nakatsuka for typing the manuscript.

This research was supported in part by National Science Foundation grants DMS 93-05601 and DMS 96-26266 and Army Research Office grant DAAH04-95-1-0105.

This manuscript was prepared using computer facilities supported in part by the National Science Foundation grants DMS 89-05292 and DMS 87-03942 awarded to the Department of Statistics at The University of Chicago, and by The University of Chicago Block Fund.

Some key words and phrases: Asymptotic Expansion, Incompleteness, Statistical uncertainty.

1. INTRODUCTION

Financial engineering relies on an approximation — that the value of stocks, bonds, currencies, etc.. follow either a continuous path or a binary tree. The methodology, going back to Black and Scholes (1973), Harrison and Kreps (1979) and Harrison and Pliskà (1981) — see, *e.g.*, Duffie (1996) for a contemporary overview — has been a powerful factor in the development of options markets. And yet, the cracks in the theory show up in phenomena such as the volatility smile (see, for example, Ch. 8.H (p. 180–185) of Duffie (1996)).

Most attempts at tackling the deviations between theory and reality have relied on creating additional flexibility in the continuous path model, in particular through stochastic volatility models (including Hull and White (1987), Ball and Roma (1994) and Duan (1995)) and implied volatility trees (Dupire (1994), Derman and Kari (1994), Rubinstein (1995)). Another line of research has abandoned continuity (or trees) altogether, and instead focused on equilibrium considerations and on approximate hedging (Merton (1976), Föllmer and Sodermann (1986), Aase (1988), Föllmer and Schweizer (1990), Schweizer (1990, 1991, 1993, 1994), Elliott and Föllmer (1991), Colwell and Elliot (1993), Mercurio and Runggaldier (1993), to mention some).

There are advantages to both these approaches. Accepting discontinuity leads to a more accurate model of prices, which typically move in multiples of fixed increments. It also permits a more useful description of shocks such as stock market crashes, currency devaluations, and similar occurrences. On the other hand, discontinuity carries with it an abandonment of the arbitrage based theory of pricing. Even in a flexible continuous path model, such pricing can be used, if need be by including coefficients representing the market price of suitable untraded securities. The advantage of continuity is pragmatic, but no less compelling for that reason.

The purpose of this paper is to present an intermediate strategy, which can draw on the strength of both the approaches described above. We shall suppose that the stock price moves in jumps of varying size, and then let the intensity of the jumps increase as their size decreases. The stock price will then converge to a geometric Brownian motion, but it will also be possible to write an asymptotic expansion for the price:

$$\text{price} \ = \ \text{Black-Scholes price} \ + \delta^{1/2}R + o(\delta^{1/2}). \tag{1.1}$$

Here, δ is a quantity which measures the distance between the actual stock price model and geometric Brownian motion.

In this setup, R is the price of a residual security in a limiting security market, and it involves observable constants and the market prices of a low dimensional set of untraded securities. To first order, therefore, $\delta^{1/2}R$ captures the cost of discontinuity, but at the same time, it is obtainable from arbitrage considerations.

It would, obviously, be more desirable to have the true price rather than an asymptotic expansion for it. (1.1), however, can be seen as an analysis of the sensitivity of the Black-Scholes price to small deviations from the standard model, and as such it should be a useful tool both theoretically and practically. – Note that compared to the results in Willinger and Taqqu (1991), Duffie and Protter (1992), Cutland, Kopp and Willinger (1993), and Amin and Khanna (1994), (1.1) provides not only a limiting result, but also the first order remainder term. In this sense, we are closer to Foster and Nelson (1996) and similar work by both these authors. The remainder term is critical in the further analysis.

In addition to providing a more accurate modeling of stocks, discontinuity also permits a realistic description of statistical uncertainty. This feature is not really available in the continuous time model. Under continuity, the volatility is observed. Hence, if it is constant there can be no statistical uncertainty (since the drift does not have an impact on the price). One can go through various attempts to induce such uncertainty by making the volatility nonconstant, but this will only recreate the problem at a meta level. Consider, for example, the case of endogenous variation. It is highly desirable to model this variation through a stochastic volatility model, say,

$$d\sigma_t^2 = \nu(\sigma_t^2)dt + \gamma(\sigma_t^2)dW_t. \tag{1.2}$$

The reason for this is that it is difficult to model mean reversion with a non random, time varying σ_t^2, and mean reversion is widely believed to be a feature of the volatility process. The only possible statistical uncertainty would then reside in ν and γ. The latter, however, is observed on the domain where σ^2 has actually varied (and is completely unknown otherwise), and ν as such does not affect the price of options, as it has to be modified by the market price of risk in σ_t^2. Also, ν is practically observable in the long run (if one incorporates information on options prices; see Renault and

Touzi (1996)). One can make ν and γ time varying, but then the question of incorporating mean reversion comes up again. And so on.

One can also model σ^2 as a function of S, or in various other ways, but as far as the author is aware, there is no known way of escaping the problem we have described in the framework of the continuous path model. One can get around it by assuming that the volatility is observed by market participants but not by econometricians (Renault (1995)), but this is not a particularly satisfying solution. One can also adopt the approach of Gottlieb and Kalay (1985), Ball (1988), and Kleive (1993) of thinking of stock prices as rounded or truncated values of a continuous process, but again this raises the question of whether stocks can be hedged continuously or not.

This is not to say that the problem of inference in models for the behavior of financial instruments is not a difficult one. A substantial literarture exists, see, *e.g.*, Bollerslev, Chou and Kroner (1992) for an overview of ARCH and GARCH models, and Dachuna-Castelle and Florens-Zmirou (1986), Bibby and Sørensen (1994, 1995), Danielsson (1994), Hansen and Scheinkman (1995), Aït-Sahalia (1996), Kessler and Sørensen (1995), Renault (1995), Foster and Nelson (1996), Gallant (1996), Hansen, Scheinkman, and Touzi (1996), Pastorello (1996) and Renault and Touzi (1996) for a selection of important recent contributions to inference in stochastic differential equations (SDEs)). It is just that there is little integration between the theory of inference and the theory of pricing.

Arguably, statistical uncertainty is not a problem — one can just use implied quantities. Things are, however, not quite so simple in the presence of many unknowns, as we shall see in Section 3. First, however, we describe the model and derive the expansion.

2. ASYMPTOTIC ANALYSIS

2.1. The Model

We suppose that $\log S_t$ is a pure jump process whose jumps have intensity λ_t, and that the jump at time t has (random) size Z_t. This can formally be carried out by letting $\log S_t$ be the integral of the Z_t process with respect to the random measure that has intensity λ_t (see, *e.g.*, Ch. II.1.d (p. 71-74) of Jacod and Shiryaev (1987)). The asymptotics are then implemented by letting the intensity go to infinity while the jump size goes to zero. To be precise, let $S_t = S_t^\delta$ be indexed by a δ which tends to zero, and suppose that the first four cumulant variations of $\log S_t^\delta$ (see Section 6.1 (p. 30-32) of Mykland (1994)) exist and have the form $\int_0^t \mu_s ds = \mu_L t + o_p(1)$, $\int_0^t \sigma_s^2 ds$, $\delta^{1/2} k_3 t + o_p(\delta^{1/2})$ and $\delta k_4 t + o_p(\delta)$. We also assume that $E \sup(\Delta \log S_t)^4 = o(\delta)$, which in particular implies that higher order optional variations of $\log S_t$ are of order $o_p(\delta)$.

The conditions assure that $\log S^\delta$ converges to a Brownian motion with drift, and the price of an option will be shown to be on the form (1.1).

Before finding the price, however, we shall investigate how to model σ_t^2.

2.2. Volatility Structure

It will be shown is Section 2.3 that the quantity R in (1.1) is the price of a derivative security in the limiting probability measure obtained when δ goes to zero. In order to obtain a meaningful expression for R, this limiting security market must have an information structure which is sensible for our purposes. This imposes requirements on the volatility structure.

First of all, σ_t^2 cannot follow (1.2), since that would lead it to be observable in the limit. In fact, if σ_L^2 is the long run average volatility, one needs $\sigma_t^2 - \sigma_L^2 = O_p(\delta^{1/2})$ to avoid this problem, because

$$[\log S, \log S]_t - \sigma_L^2 t = \int_0^t (\sigma_u^2 - \sigma_L^2) du + O_p(\delta^{1/2}), \tag{2.1}$$

so otherwise, σ_t^2 would be observable in the limit. We shall therefore suppose that

$$\sigma_t^2 = \sigma_L^2 + \delta^{1/2} \eta_t^{(\sigma)} + o_p(\delta^{1/2}). \tag{2.2}$$

For similar reasons, we let $\lambda_L \delta^{-1}$ be the long run average intensity process. It is natural to suppose that

$$\lambda_t = \delta^{-1} \lambda_L + \delta^{-1/2} \lambda_L \eta_t^{(\lambda)} + o_p(\delta^{1/2}). \tag{2.3}$$

(One could also work with the case where λ_t/δ tends to zero or infinity, but this is less interesting as one factor will then dominate).

To accomplish (2.2)–(2.3), one can suppose that

$$d(\sigma_t^2, \delta\lambda_t)^* = \nu(\sigma_t^2, \delta\lambda_t)dt + \delta^{1/2} \gamma(\sigma_t^2, \delta\lambda_t) dW_t. \tag{2.4}$$

$\eta_t = (\eta_t^{(\sigma)}, \eta_t^{(\lambda)})^*$ then takes the form

$$d\eta_t = \nu' \eta_t dt + \gamma dW_t, \tag{2.5}$$

where $\nu' = \nu'(\sigma_L^2, \lambda_L)$ and $\gamma = \gamma(\sigma_L^2, \lambda_L)$, and where it is assumed that $\nu(\sigma_L^2, \lambda_L) = 0$. Since correlation between the two components of the Brownian motion W can be moved into the γ, we shall suppose these components to be independent standard Brownian motions. Note that (η_t) is mean reverting if $\det(\gamma) \neq 0$ (at (σ_L, λ_L)) and ν' is negative definite (in the sense that the eigenvalues λ of ν' satisfy $\mathrm{Re}(\lambda) < 0$).

The principal auxiliary processes in our system are then the observed standardized square variation

$$\xi_t^\delta \;=\; \delta^{-1/2}([\log S, \log S]_t - \sigma_L^2 t) \tag{2.6}$$

and the observed standardized activity level

$$n_t^\delta \;=\; \delta^{1/2}(\#\text{ jumps from 0 to } t - \delta^{-1}\lambda_L t) \tag{2.7}$$

These quantities are 'observed' on the principle that to first approximation, long run quantities like μ_L, σ_L^2 and λ_L are observable.

Set

$$M = \begin{bmatrix} k_4 - \left(\frac{k_3}{\sigma_L}\right)^2 & \sigma_L^2 \\ \sigma_L^2 & \lambda_L \end{bmatrix}. \tag{2.8}$$

One then gets the limiting behavior of our observables.

THEOREM 1. *Assume the conditions of Section 2.1. Also assume (2.4), that γ is continuous and that its eigenvalues are bounded away from zero and infinity, and that ν is continuously differentiable. Then M is nonnegative definite, and (2.2)–(2.3) holds with respect to the supremum distance and with η given by (2.5). Furthermore, $(S^\delta, \xi^\delta, n^\delta, W)$ converge jointly in law to (S, ξ, n, W), where*

$$dS_t = \left(\mu_L + \frac{\sigma_L^2}{2}\right) S_t dt + \sigma_L S_t dB_t, \tag{2.9}$$

$$\xi_t = \tilde{\xi}_t + \frac{k_3}{\sigma_L} B_t \tag{2.10}$$

and

$$(\widetilde{\xi}_t, n_t)^* = \int_0^t \eta_u du + M^{1/2} V_t, \tag{2.11}$$

where $M^{1/2}$ *is any symmetric square root of* M, $V_t = (V_t^{(1)}, V_t^{(2)})^*$, *and* $(B_t, V_t^{(1)}, V_t^{(2)}, W_t^{(1)}, W_t^{(2)})$ *is a standard Brownian motion. The convergence in law is in the space of càdlàg functions on* $[0,T]$ *with respect to the supremum distance.*

PROOF. See the Appendix.

2.3. The Price of Options

Suppose we wish to hedge a European option which expires at time T, and for simplicity assume that the risk free interest interest rate r is constant. If $C(S,t,\sigma)$ is the Black-Scholes price. a natural hedging strategy is to use a "delta" of the form $\dot{C}_s(S_{t-}^\delta, t, \hat{\sigma}_{t-}^\delta)$. Set

$$X_t^{\delta *} = C(S_0, 0, \sigma_L) + \int_0^t \dot{C}_S(S_{u-}^\delta, u, \hat{\sigma}_{u-}^\delta) dS_u^{\delta *}, \tag{2.12}$$

where $S_t^{\delta *} = e^{-rt} S_t^\delta$. If we let $X_t^\delta = e^{rt} X_t^{\delta *}$, then by numeraire invariance, there is a self-financing strategy for X_T^δ, the price of which is $C(S_0, 0, \sigma_L)$.

We wish to evaluate the residual exposure to risk after approximately hedging $C(S_T^\delta, T)$ with X_T^δ. Set

$$R_t^\delta = \delta^{-1/2} \left(C^*(S_t^\delta, t, \sigma_L) - X_t^{\delta *} \right), \tag{2.13}$$

where $C^* = e^{-rt} C$. To calculate the residual price R in (1.1), we find the limit R_t of R_t^δ, and from this,

$$R = E^*(R_T \mid \mathcal{F}_0), \tag{2.14}$$

where P^* is a risk neutral probability to be discussed (cf. Harrison and Kreps (1979)).

Before we proceed any further, note that R is independent of hedging strategy $\hat{\sigma}_t^\delta$, since the price of X_T^δ is independent of this choice. For the purpose of evaluating R, therefore, we shall. without loss of generality, set

$$\hat{\sigma}_t^\delta = \sigma_L. \tag{2.15}$$

We shall use $C(S,t) = C(S,t,\sigma_L)$. This does not mean, however, that the level of uncertainty about σ_t does not have an impact on the price R, as we shall see in the following. First, however, the limit result.

THEOREM 2. *Assume the conditions of Theorem 1, and also that* $\dddot{C}_{SSS}(S,t)$ *exists and is continuous, uniformly in* t, *for* $S \in A$, *where* $\mathbb{R} - A$ *has arbitrarily small Lebesgue measure. Then,* (R_t^δ) *converges in law, jointly with and in the same topology as the processes in Theorem 1, to*

$$R_t = \tfrac{1}{2} \int_0^t S_u^2 \ddot{C}^*_{SS}(S_u,u) d\xi_u + \frac{1}{3!} k_3 \int_0^t f(S_u,u) du, \tag{2.16}$$

where

$$f(S,u) = 3S^2 \ddot{C}^*_{SS}(S,u) + S^3 \dddot{C}^*_{SSS}(S,u). \tag{2.17}$$

Note that the conditions of Theorem 2 are trivially satisfied for call and put options.

Returning now to the price of the residual, note that

$$R = \frac{1}{3!} k_3 T \left\{ 3 \left(1 - \frac{\mu_L - r}{\sigma_L^2} \right) S_0^2 \ddot{C}_{SS}(S_0,0) + S_0^3 \dddot{C}_{SSS}(S_0,0) \right\} + R' \tag{2.18}$$

where

$$R' = \tfrac{1}{2} E^* \left(\int_0^T S_u^2 \ddot{C}^*_{SS}(S_u,u) d\tilde{\xi}_u \mid \mathcal{F}_0 \right). \tag{2.19}$$

This is because

$$E^* \left(S_t^p \frac{\partial^p}{\partial S^p} C^*(S_t,t) \mid \mathcal{F}_0 \right) = S_0^p \frac{\partial^p}{\partial S^p} C^*(S_0,0), \tag{2.20}$$

and because, by (2.10),

$$\xi_t = \tilde{\xi}_t + \frac{k_3}{\sigma_L} \left(B_t^* - \frac{\mu_L - r}{\sigma_L} t \right), \tag{2.21}$$

where B^* is a Brownian motion under P^*.

To find R', we need the value process of $\tilde{\xi}_t$. In the most general form, suppose that v_t and w_t are the (two-dimensional) market prices of V_t and W_t, respectively, so that $V_t - v_t$ and $W_t - w_t$

are P^*-martingales. It is then easy to see that

$$R' = \tfrac{1}{2}E^*\left(\int_0^T S_u^2 \ddot{C}^*_{SS}(S_u,u)\left(\tilde{\eta}_u^{(\sigma)} + (1,0)M^{1/2}v_u\right) du \mid \mathcal{F}_0\right), \tag{2.22}$$

where $\tilde{\eta}_t$ is the solution of

$$\begin{aligned} d\tilde{\eta}_t &= \nu'\tilde{\eta}_t dt + \gamma w_t dt \\ \tilde{\eta}_0 &= \eta_0. \end{aligned} \tag{2.23}$$

A substantial simplification is obtained by assuming either that (v_t) and (w_t) are independent of (S_t), or that they follow a linear differential equation

$$d(v_t, w_t)^* = A_0(t)dt + A_1(t)(v_t, w_t)^* dt + A_2(t)dZ_t, \tag{2.24}$$

where the A's are nonrandom and where Z_t can have B_t as one component, with the other components independent of (B_t). In this case, it is straightforward to see that

$$R' = \tfrac{1}{2}S_0^2\ddot{C}_{SS}(S_0,0)E^*(\tilde{\xi}_T \mid \mathcal{F}_0). \tag{2.25}$$

In this case, since

$$\dot{C}^*_\sigma(S,t,\sigma) = (T-t)\ddot{C}^*_{SS}(S,t,\sigma)S^2, \tag{2.26}$$

one can alternatively write (1.1) as

$$\text{price} = C(S_0,0,\sigma_h) + \delta^{1/2}\frac{1}{3!}k_3 T S_0^3 \dddot{C}_{SSS}(S_0,0,\sigma_L) + o(\delta^{1/2}), \tag{2.27}$$

where

$$\sigma_h^2 = \sigma_L^2 + \delta^{1/2}\tfrac{1}{2}\left(k_3\left(1 - \frac{\mu_L - r}{\sigma_L^2}\right) + \frac{1}{T}E^*\left(\tilde{\xi}_T \mid \mathcal{F}_0\right)\right). \tag{2.28}$$

Though the given solution features skewness (as in k_3) more prominently than kurtosis, the latter is present through k_4 in the matrix M (in (2.8)). If one were to go to a second order expansion, k_4 would also be present in other ways.

3. ESTIMATION

3.1. Implied Quantities

Three cases can occur:

i) There are exchange traded options, and one wishes to price an option with the same maturity as an exchange traded one.

ii) There are exchange traded options, and one wishes to price an option with a different maturity than the exchange traded ones.

iii) There are no exchange traded options on the security S.

Reality, obviously, is more complex; the exchange traded options could be American, or one might wish to price non European options. For simplicity, however, we continue to concentrate on European options.

Case (i) is the simplest — one can find $E^*(\xi_T \mid \mathcal{F}_0)$ by regression of the exchange traded options. Or one can use (2.27) to find σ_h for each strike price after correcting for the second term on the r.h.s. of (2.27). If the approximation is good, this yields a fairly flat curve for σ_h as a function of the strike price. One should not expect a perfect fit, for there can be error terms of order $O(\delta)$. How to carry out the regression or the averaging of the σ_h's (scale? weights?) is an interesting question, similar to issues already debated in connection with implied volatilities. We shall, however, not go into this issue here.

Case (iii) means that one has no data with which to determine the relevant market prices of risk. In this case, one's best bet would probably be to use considerations along the lines of Schweizer (1992) and Pham and Touzi (1996).

Case (ii) is the most interesting one from our perspective. One has observations (or estimates) of $E^*(\xi_T \mid \mathcal{F}_0)$ for a couple of expiration dates T, and one wishes to interpolate between them. There are two strategies:

- using implied quantities only, or
- using a combination of statistical estimates and implied quantities.

One can also draw on equilibrium considerations, but we shall not consider this angle here.

Relying only on implied quantities is at best unwise, and often impossible. Even in the benign case where the A_i's in (2.24) are independent of t, and hence approximately observed from long run data, one gets that

$$E^*(\xi_T \mid \mathcal{F}_0) = f(\eta_0, v_0, w_0, T) \tag{3.1}$$

for some function f. Since each of η_0, v_0 and w_0 are two dimensional and unobserved, f is a function of three to six unknowns, depending on the matrices specifying the system.

This means that one needs three to six different option expiration dates to infer the quantities needed. (Different strike prices for the same maturity do not give additional information for the model). Many options do not even have that many maturities. Even if they do, this is a dangerous reliance on the validity of the model. It would be safer to use additional observation points T to assess the correctness of the predictions of the model.

For this reason, we shall in the following take a statistical approach to reducing the number of implied quantities needed.

3.2. Uncertainty Equilibrium

Our limiting system consist of observed processes S, ξ and n, and unobserved processes η and V. It seems natural to assume that one's uncertainty is in equilibrium, in the sense that for all t,

$$\Gamma_t \;=\; \Gamma, \tag{3.2}$$

where Γ_t is the posterior covariance matrix for η_t given the data up to t. Alternatively, one could assume that

$$\text{posterior c.d.f. of } \eta_t \text{ at } t \;=\; F(x - \hat{\eta}_t), \tag{3.3}$$

where F is fixed and where $\hat{\eta}_t$ is the (vector) posterior. The reason why we propose this is that one is unlikely to systematically gain or lose information about the unobserved state variables over time. In other words, an investor will typically not know more about the current volatility on March 22 than on January 7. One could, obviously, model Γ or F as mean reverting processes. We avoid this here in order to keep the model as simple as possible.

Uncertainty equilibrium has implications for what prior and posterior distributions must look like.

THEOREM 3. *Assume the conditions of Theorem 1. Also suppose that ν' is negative definite and that M is positive definite. Then the equation*

$$\Gamma M^{-1}\Gamma - \nu'\Gamma - \Gamma(\nu')^* - \gamma\gamma^* = 0, \tag{3.4}$$

has a unique symmetric positive definite solution. Furthermore the following statements are equivalent:

(i) (3.2), and the distribution on η_0 given the data at time 0 is Gaussian: and

(ii) (3.3), and F is absolutely continuous with full support on R^2.

Under either of these equivalent assumptions, the covariance matrix is the symmetric positive definite solution of (3.4). Also, the conditional expectations satisfy

$$d\hat{\eta}_t = \nu'\hat{\eta}_t dt + \Gamma M^{-1}(d(\widetilde{\xi}_t, n_t)^* - \hat{\eta}_t dt). \tag{3.5}$$

PROOF: See the Appendix. Note that a reasonably explicit solution to (3.4) is constructed in the proof.

In other words, uncertainty equilibrium imposes the prior variance. and, if one assumes (3.6), the entire prior distribution. Under these circumstances, Bayesian inference would appear to be a reasonable choice for characterizing the available information.

In the above, note that since μ_L, σ_L^2 and k_3 are long run quantities which are assumed to be observed, and $\widetilde{\xi}_t$ is an observed process. It is possible to take $\sigma_t^2 = \sigma_L^2$, but we then suppose the volatility to be observed. Otherwise, the assumption of uncertainty equilibrium would be violated.

Now set

$$\hat{V}_t = M^{-1/2}\left(\left(\widetilde{\xi}_t, n_t\right)^* - \int_0^t \hat{\eta}_u du\right). \tag{3.6}$$

Since $\hat{\eta}_t$ is a conditional expectation given the data, it follows that $\hat{V}_t$ is a martingale in the filtration generated by the observables. Since the quadratic variation structure of $(B, \hat{V})$ is the same as for

(B, V), Levy's Theorem (see, *e.g.*, Theorem II.4.4 (p. 102) of Jacod and Shiryaev (1987)) yields that $(B, \hat{V})$ is a standard Wiener process with respect to the observable filtration.

It follows from Theorem 3 that

$$d\hat{\eta}_t = \nu' \hat{\eta}_t dt + \Gamma M^{-1/2} d\hat{V}_t. \tag{3.7}$$

Hence, the observed system is characterized by (2.9)–(2.10), (3.7), and

$$(\widetilde{\xi}_t, n_t)^* = \int_0^t \hat{\eta}_u du + M^{1/2} \hat{V}_t. \tag{3.8}$$

3.3. The Effect of Uncertainty on Prices

Formula (2.19) continues to hold, but we now have a new observed Doob-Meyer decomposition for $\widetilde{\xi}_t$: the one given by (3.8). The market now only has two Brownian motions rather than four, and if one takes $d\hat{V}_t - \hat{v}_t dt$ to be a martingale P^*, one gets that (2.22) remains valid with

$$v_t = \hat{v}_t, \tag{3.9}$$

but $\widetilde{\eta}_t$ now is the solution of

$$\begin{aligned} d\widetilde{\eta}_t &= \nu' \widetilde{\eta}_t dt + \Gamma M^{-1/2} \hat{v}_t dt \\ \widetilde{\eta}_0 &= \hat{\eta}_0. \end{aligned} \tag{3.10}$$

In other words γw_t gets replaced by $\Gamma M^{-1/2} \hat{v}_t$, and η_0 by $\hat{\eta}_0$.

If we specialize to the case (2.25), and assume that the market price $\hat{v}_t$ follows a mean reverting process with constant coefficients, then

$$E(\xi_T \mid \mathcal{F}_0) = f(\hat{\eta}_0, \hat{v}_0, T) \tag{3.11}$$

where $\hat{\eta}_0$ is observed and $\hat{v}_0$ still needs to be inferred from traded options. This is only a two-dimensional quantity, however, so only two expiration dates are needed. Note that $\mathcal{F}$ now represents the observed filtration.

For an explicit expression for f, suppose that

$$\hat{v}_t = \hat{v}_L + \hat{\hat{v}}_t, \tag{3.12}$$

where $\hat{v}_L$ is the long run value of $\hat{v}_t$. Also assume that $\hat{v}_t$ follows a mean reverting process

$$d\hat{v}_t = A\hat{v}_t dt + A_0 dZ_t. \tag{3.13}$$

Then

$$\begin{aligned} f(\hat{\eta}_0, \hat{v}_0, t) = {} & (1,0)(\nu')^{-1}e^{\nu'}\hat{\eta}_0 \\ & + (1,0)E^* \left((\nu')^{-1} \int_0^t e^{\nu'(t-s)} \Gamma M^{-1/2} \hat{v}_S ds \right. \\ & \left. -(\nu')^{-1} \int_0^t \Gamma M^{-1/2} \hat{v}_s ds + \int_0^t M^{1/2} \hat{v}_s ds \mid \mathcal{F}_0 \right), \end{aligned} \tag{3.14}$$

and so

$$\begin{aligned} & f(\hat{\eta}_0, \hat{v}_0, t) \\ & = (1,0)(\nu')^{-1}e^{\nu' t}\hat{\eta}_0 \\ & + (1,0) \left[(\nu')^{-2}(e^{\nu' t} - \nu' t - I)\Gamma M^{-1/2} - M^{1/2} t \right] \hat{v}_L \\ & + (1,0) \left[\Omega_t + \left(M^{1/2} - (\nu')^{-1}\Gamma M^{-1/2} \right) A^{-1}(e^{At} - I) \right] (\hat{v}_0 - \hat{v}_L) \end{aligned} \tag{3.15}$$

where Ω_t is the solution of

$$\nu'\Omega_t - \Omega_t A = (\nu')^{-1}\Gamma M^{-1/2} e^{At} + (\nu')^{-1} e^{\nu' t} \Gamma M^{-1/2} \,. \tag{3.16}$$

This jolly little expression is, of course, more suitable for software than for human consumption, but the point is that all the quantities in (3.15), except for $\hat{\eta}_0$ and $\hat{v}_0$, are practically observable in the long run. $\hat{\eta}_0$ is observed, and one only needs, therefore, two implied quantities, namely the two components of $\hat{v}_0$. This will use two expiration dates to infer the relevant quantities. If there are more dates available, one can use regression, and one can also do model checking.

How is $\hat{\eta}_0$ computed? For the purpose of our limiting argument, any predictor $\hat{\eta}_0^\delta$ which converges in probability to $\hat{\eta}_0$ as δ tends to zero will do. Further fine tuning, as in Florens-Zmirou (1993), Chesney, Elliot, Madan and Yang (1993) and Pastorello (1996), is obviously desirable in practice. though we shall not pursue this issue here.

4. CONCLUSION

We have seen in the above that asymptotic methods can be effective in studying incomplete markets. Also, it is apparent that it is undesirable to rely on implied quantities only, and that one needs to also use historical data to extract information about unobserved quantities. The effect of estimation is incorporated into the options price in a natural manner.

The specific model and the specific type of asymptotics used is by no means meant to be anything more than a test case and an example. There is room for substantial exploration into how one can best implement the methodology advocated here.

APPENDIX

PROOF OF THEOREM 1. Write

$$\log S_t^\delta = \log S_0 + \int_0^t \mu_u du + L_t^S$$

$$\xi_t^\delta = \delta^{-1/2} \int_0^t (\sigma_u^2 - \sigma_L^2) du + L_t^\xi$$

and

$$n_t^\delta = \delta^{1/2} \int_0^t (\lambda_u - \delta^{-1}\lambda_L) du + L_t^n.$$

$L = (L^S, L^\xi, L^n)$ is then a martingale with predictable covariation matrix given by

$$\langle L, L\rangle_t = \begin{bmatrix} \langle L^S, L^S\rangle_t & & \\ \delta^{-1/2}\langle L^S, L^S, L^S\rangle_t & \delta^{-1}\langle L^S, L^S, L^S, L^S\rangle_t & \\ \delta^{1/2}\int_0^t \mu_s ds & \langle L^S, L^S\rangle_t & \delta\int_0^t \lambda_s ds \end{bmatrix}$$
$$= \begin{bmatrix} \sigma_L^2 t & k_3 t & 0 \\ k_3 t & k_4 t & \sigma_L^2 t \\ 0 & \sigma_L^2 t & \lambda_L t \end{bmatrix} + o_p(1) \tag{A.1}$$

Since $E \sup(\Delta L_t^S)^4 = o(\delta)$, and since L and W have zero covariation, it follows from Theorem VIII.3.12 (p. 432–433) in Jacod and Shiryaev (1987) that (L, W) converges in law to a Brownian motion, with L independent of W and with covariance matrix given as the limit in (A.1). The result then follows since $\int_0^t \delta^{-1/2}(\sigma_s^2 - \sigma_L^2)ds$ and $\int_0^t \delta^{1/2}(\lambda_s - \delta^{-1}\lambda_L)ds$ converge in probability to the integrals of the respective limits, by the uniformity in (2.2)–(2.3). *Q.E.D.*

PROOF OF THEOREM 2. By Ito's Lemma (see, *e.g.*, Theorem I.4.57 (p. 57) of Jacod and Shiryaev (1987)),

$$\begin{aligned} \delta^{1/2} dR_t^\delta &= \Delta C^*(S_t^\delta, t) - \dot{C}_S^*(S_{t-}^\delta, t)\Delta S_t^\delta \\ &- \quad \tfrac{1}{2}\ddot{C}_{SS}^*(S_{t-}^\delta, t)(S_{t-}^\delta)^2 \sigma_L^2 dt \\ &= \Delta C^*(S_t^\delta, t) - \dot{C}_{\log S}^*(S_{t-}^\delta, t)\frac{\Delta S_t^\delta}{S_{t-}} \\ &\quad - \tfrac{1}{2}\left[\ddot{C}_{\log S, \log S}^*(S_{t-}^\delta, t) - \dot{C}_{\log S}^*(S_{t-}^\delta, t)\right]\sigma_L^2 dt, \end{aligned} \tag{A.2}$$

where we have used that $\dot{C}_t^* + r\dot{C}_S^* S + \frac{1}{2}\ddot{C}_{SS}^* S^2\sigma_L^2 = 0$. The dependence on σ_L^2 is suppressed in the notation. By Taylor expansion,

$$\begin{aligned} R_t^\delta &= \tfrac{1}{2}\int_0^t \ddot{C}_{SS}^*(S_{u-}^\delta, u)d\xi_u^\delta \\ &\quad + \frac{1}{3!}\delta^{-1/2}\int_0^t f(Z_u^\delta, u)d[\log S^\delta, \log S^\delta, \log S^\delta]_u \end{aligned} \tag{A.3}$$

where $|Z_t^\delta - S_t^\delta| \leq |\Delta S_t^\delta|$. Z is measurable and càdlàg by construction, and by Lemma VI.3.31 (p. 316) of Jacod and Shiryaev (1987) and (our) Theorem 1, Z^δ converges to S in law. By the continuity assumption of the current theorem, $f(Z_\cdot^\delta, \cdot)$ converges to $f(S_\cdot, \cdot)$ in law (Remark VI.3.8 (p. 312) in Jacod and Shiryaev (1987)). One can then use Theorem VI.3.21 (i) (p. 314) in the same work together with uniform integrability to conclude that the second term on the r.h.s. of (A.3) converges as specified to the second term on the r.h.s. of (2.16). This is because $\delta^{-1/2}[\log S^\delta, \log S^\delta, \log S^\delta]_t = [\xi^\delta, \log S^\delta]_t$, the convergence properties of which follow from Corollary VI.6.7 (p. 342) in Jacod and Shiryaev, in view of (our) Theorem 1.

The first term on the r.h.s. of (A.3) is handled in analogy to the proof of Theorem 1. *Q.E.D.*

PROOF OF THEOREM 3—SOLUTION OF EQUATION (3.4). Since the solution is obvious if ν' is symmetric, we shall exclude this case in the following.

Set $\widetilde{\Gamma} = M^{-1/2}\Gamma M^{-1/2}$, $\widetilde{\nu}' = M^{-1/2}\nu' M^{-1/2}$ and $\widetilde{\gamma} = M^{1/2}\gamma$. The problem is equivalent, obviously, to finding a unique symmetric nonnegative definite (u.s.n.d.) solution to

$$(\widetilde{\Gamma} - \widetilde{\nu}')(\widetilde{\Gamma} - \widetilde{\nu}')^* = \widetilde{\nu}'(\widetilde{\nu}')^* + \widetilde{\gamma}\widetilde{\gamma}^*. \tag{A.4}$$

If one writes the right hand side of (A.4) as VLV^* where L is diagonal and V is orthonormal, one can further set $\bar{\Gamma} = V^*\widetilde{\Gamma}V$ and $\bar{\nu}' = V^*\widetilde{\nu}'V$, and the problem then reduces to finding a u.s.n.d. solution to

$$(\bar{\Gamma} - \bar{\nu}')(\bar{\Gamma} - \bar{\nu}')^* = L. \tag{A.5}$$

The set of $\bar{\Gamma}$ satisfying (A.5) with no further constraint is given by

$$\bar{\Gamma} = L^{1/2}U + \bar{\nu}' \tag{A.6}$$

where U can be any orthonormal matrix.

Since $\bar{\Gamma}$ must be symmetric, the two off diagonal elements of $L^{1/2}U + \bar{\nu}'$ must be the same, so

$$\ell_2^{1/2} u_{21} - \ell_1^{1/2} u_{12} = \bar{\nu}'_{12} - \bar{\nu}'_{21}. \tag{A.7}$$

Also, since U is orthogonal,

$$u_{11}u_{21} + u_{22}u_{12} = 0. \tag{A.8}$$

Solving these two equations yield that

$$\begin{aligned} u_{21} &= u_{22}(\bar{\nu}'_{12} - \bar{\nu}_{21})/\det \\ u_{12} &= -u_{11}(\bar{\nu}_{12} - \bar{\nu}_{21})/\det, \end{aligned} \tag{A.9}$$

where

$$\det = \ell_1^{1/2} u_{11} + \ell_2^{1/2} u_{22} \tag{A.10}$$

(we discuss below why $\det \neq 0$). $u_{11}^2 + u_{12}^2 = 1$ is equivalent to

$$u_{11}^2 \left(1 + \frac{(\bar{\nu}'_{12} - \bar{\nu}'_{21})^2}{\det^2} \right) = 1, \tag{A.11}$$

and, similarly, $u_{21}^2 + u_{22}^2 = 1$ to

$$u_{22}^2 \left(1 + \frac{(\bar{\nu}'_{12} - \bar{\nu}'_{21})^2}{\det^2} \right) = 1. \tag{A.12}$$

Hence $u_{11} = \pm u_{22}$. The minus sign is ruled out. however, since otherwise (by (A.9)) U would be symmetric and hence (since orthonormal) the identity. This could mean that $\bar{\nu}'_{12} = \bar{\nu}'_{21}$, whence ν' would be symmetric, which we have previously ruled out.

A solution exists, therefore, provided (A.11) and $\det \neq 0$ (if $\det = 0$, there is no solution, since the case ν' symmetric has been ruled out). This is equivalent to

$$|\bar{\nu}'_{12} - \bar{\nu}'_{21}| < \ell_1^{1/2} + \ell_2^{1/2}. \tag{A.13}$$

This condition is satisfied since

$$(\bar{\nu}'_{12} - \bar{\nu}'_{21})^2 \leq \operatorname{tr}(\bar{\nu}'\bar{\nu}'^{*})$$

$$
\begin{aligned}
&< \operatorname{tr}(L) \\
&< (\ell_1^{1/2} + \ell_2^{1/2}). \qquad \text{(A.14)}
\end{aligned}
$$

There are now two possible solutions of our system, corresponding to positive or negative sign of u_{11}. However,

$$\operatorname{tr}(\bar{\Gamma}) = u_{11}(\ell_1^{1/2} + \ell_2^{1/2}) + \operatorname{tr}(\bar{\nu}') \tag{A.15}$$

which is negative if $u_{11} < 0$. This contradicts the requirement that $\bar{\Gamma}$ be positive definite. Hence, the only possible solution for U is

$$U = \begin{pmatrix} (1-\alpha^2)^{1/2} & -\alpha \\ \alpha & (1-\alpha^2)^{1/2} \end{pmatrix} \tag{A.16}$$

where

$$\alpha = \frac{\bar{\nu}'_{12} - \bar{\nu}'_{21}}{\ell_1^{1/2} + \ell_2^{1/2}}. \tag{A.17}$$

This solution exists (*i.e.*, $|\alpha| < 1$), Γ satisfies (3.4), and is symmetric. It is also the only solution which may be positive definite. To show that this requirement is indeed satisfied, with $\widetilde{\Gamma} = Q\Lambda Q^*$, and note that Λ can be chosen so that $\widetilde{\Gamma}$ is positive definite, and hence this must be a property of the solution we have described. *Q.E.D.*

PROOF OF THE REMAINDER OF THEOREM 3. Provided (i) holds, it follows from Theorem 12.7 (p. 33) of Lipster and Shiryaev (1978) that Γ is the solution of (3.4). Equation (3.5) is a consequence of the same result.

(i) ⇒ (ii) follows from the above and from Theorem 12.6 (p. 31) of Lipster and Shiryaev (1978).

(ii) ⇒ (i): Let P be the probability distribution, and let Q be one in (i). By our assumptions, dP/dQ exists and is $\mathcal{F}_0$-measurable. Since ν' is negative definite, (η_t) is ergodic, and so η_t is asymptotically independent of $\mathcal{F}_0$ and hence dP/dQ. It follows that, for bounded continuous g,

$$
\begin{aligned}
&E\left(g(\eta_t) \mid \mathcal{F}_t^{\text{obs}}\right) \\
&\quad = \frac{E_Q\left(g(\eta_t)\frac{dP}{dQ} \,\middle|\, \mathcal{F}_t^{\text{obs}}\right)}{E_Q\left(\frac{dP}{dQ} \,\middle|\, \mathcal{F}_t^{\text{obs}}\right)}
\end{aligned}
$$

$$= E_Q \left(g(\eta_t) \mid \mathcal{F}_t^{\text{obs}} \right) + o_p(1).$$

Hence, by (ii), we must have $Q = P$, and (i) follows. *Q.E.D.*

REFERENCES

Aase, K. K. (1988). Contingent claims valuation when the security price is a combination of an Ito process and a random point process, *Stoc. Proc. Appl.* **28** 185- 220.

Aït-Sahalia, Y. (1996). Nonparametric Pricing of Interest Rate Derivative Securities. *Econometrica* **64** 527-560.

Amin, K., and Khanna, A. (1994). Convergence of American option values from discrete- to continuous-time financial models, *Math. Finance* **4** 289–304.

Ball, C.A. (1988). Estimation bias induced by discrete security prices. *J. Finance* **4** 841-865.

Ball, C.A., and Roma, A. (1994). Stochastic volatility option pricing, *J. Financial and Quantitative Analysis* **29** 589-607.

Bibby, B.M., and Sørensen, M. (1994). Martingale estimation functions for discretely observed diffusion processes. Technical report no. 277, Dept. of Theoretical Statistics, University of Aarhus.

Bibby, B.M., and Sørensen, M. (1995). On estimation for discretely observed diffusions: a review. Technical report no. 334, Dept. of Theoretical Statistics, University of Aarhus.

Black, F., and Scholes, M. (1973). The pricing of options and corporate liabilities, *J. Polit. Econ.* **81** 637-654.

Bollerslev, T., Chou, R.Y., and Kroner. K.F. (1992). ARCH modeling in Finance, *J. Econometrics* **52** 5-59.

Chesney, M., Elliot, R.J., Madan, D., and Yang, Y. (1993). Diffusion coefficient estimation and asset pricing when risk premia and sensitivities are time varying, *Math. Finance* **3** 85-99.

Colwell, D.B., and Elliot, R.J. (1993). Discontinuous asset prices and non-attainable contingent claims, *Math. Finance* **3** 295-308.

Cutland, N.J., Kopp, E., and Willinger, W. (1993). From discrete to continuous financial models: new convergence results from option pricing, *Math. Finance* **3** 101–123.

Dacunha-Castelle, D. & Florens-Zmirou, D. (1986). Estimation of the coefficients of a diffusion from discrete observations, *Stochastics* **19** 263–284.

Danielsson. J. (1994). Stochastic volatility in asset prices: Estimation with simulated maximum likelihood, *J. Econometrics* **64** 375-400.

Derman. E., and Kani, I. (1994). Riding on the smile, *Risk* **7** (February) 32-39.

Duan, J.-C. (1995). The Garch option pricing model, *Math. Finance* **5** 13-32.

Duffie, D. (1996). *Dynamic Asset Pricing Theory* (2nd ed.) (Princeton).

Duffie, D., and Protter, P. (1992). From discrete- to continuous-time finance: weak converngence of the financial gain process, *Math. Finance* **2** 1–15.

Dupire, B. (1994). Pricing with a smile, *Risk* **7** (January) 18-20.

Elliot, R.J., and Föllmer, H. (1991). Orthogonal martingale representation, in *Stochastic Analysis*, eds. E. Mayer-Wolf, E. Merzbach, and A. Schwartz (Academic Press).

Florens-Zmirou, D. (1993). On estimating the diffusion coefficient from discrete observations, *J. Appl. Prob.* **30** 790-804.

Föllmer. H. and Schweizer, M. (1991). Hedging of Contingent Claims under Incomplete Information, in in *Applied Stochastic Analysis*. eds. M.H.A. Davis and R.J. Elliot (Gordon and Breach).

Föllmer, H. and Sondermann, D. (1986). Hedging of non-redundant contingent claims. in Hildebrand, W. and Mas-Colell, A. (eds) *Contributions to Mathematical Economics*. p. 205-223.

Foster, D.P., and Nelson, D.B. (1996). Continuous record asymptotics for rolling sample variance estimators, *Econometrica* **64** 139-74.

Gallant. A.R. (1996). Efficient method of moments with application to estimating stochastic differential equations (preprint).

Gottlieb. G., and Kalay, A. (1985). Implications of the discreteness of observed stock prices. *J. Finance* **1** 135-153.

Hansen, L.P., and Scheinkman, J.A. (1995). Back to the future: Generating moment implications for continuous-time Markov processes, *Econometrica* **63** 767-804.

Hansen. L.P., Scheinkman, J.A., and Touzi, N. (1996). Identification and imbeddability of Markov diffusion models from discrete time data. preprint.

Harrison, J.M., and Kreps, D.M. (1979). Martingales and arbitrage in multiperiod securities markets, *J. Econ. Theory* **20** 381-408.

Harrison, J. M., and Pliska, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading, *Stoc. Proc. Appl.* **11** 215- 260.

Hull, J.C., and White, A. (1987). The pricing of options on assets with stochastic volatilities, *J. Finance* **42** 281-300.

Jacod, J., and Shiryaev, A. N. (1987). *Limit Theory for Stochastic Processes*, (Springer-Verlag).

Jacquier, E., Polson, N. and Rossi, P.E. (1994). Bayesian analysis of stochastic volatility models *J. Bus. Econ. Statist.* **12** 371–389.

Kessler, M., and Sørensen, M. (1995). Estimating equations based on eigenfunctions for a discretely observed diffusion process, Technical report no. 332, Dept. of Theoretical Statistics, University of Aarhus.

Kleive, A. M. (1993). *Estimering av stokastiske differensialligninger med vekt på effekten av grupperte data* (Dept. of Mathematics, University of Bergen).

Liptser, R.S., and Shiryayev, A.N. (1978). *Statistics of Random Processes* II (Springer-Verlag).

Mercurio, F., and Runggaldier. W.J. (1993). Option pricing for jump diffusions: approximations and their interpretation, *Math. Finance* **3** 191-200.

Merton, R.C. (1976). Option pricing when underlying stock returns are discontinuous, *J. Financial Econ.* **3** 125-144.

Mykland, P.A. (1994). Bartlett type identities for martingales. *Ann. Statist.* **22** 21–38.

Pastorello, S. (1996). Diffusion coefficient estimation and asset pricing when risk premia and sensitivities are time varying: A comment, *Math. Finance* **6** 111-17.

Pham, H. and Touzi, N. (1996). Intertemporal equilibrium risk premia in a stochastic volatility model, *Math. Finance* **6** 215-236.

Renault. E. (1995). Econometric models of option pricing errors (preprint).

Renault, E., and Touzi, N. (1996). Option hedging and implicit volatilities in a stochastic volatility model, *Math. Finance* **6**, 279-302.

Rubinstein, M. (1994). Implied binomial trees, *J. Finance* **49** 771-818.

Schweizer, M. (1990). Risk-minimality and orthogonality of martingales *Stochastics* **30** 123-131.

Schweizer, M. (1991). Option hedging for semimartingales *Stoch. Proc. Appl.* **37** 339- 363.

Schweizer, M. (1992). Mean-variance hedging for general claims *Ann. Appl. Prob.* **2** 171-179.

Schweizer, M. (1993). Semimartingales and hedging in incomplete markets *Th. Prob. Appl.* **37** 169- 171.

Schweizer, M. (1994). Approximating random variables by stochastic integrals, *Ann. Probab.* **22** 1536-1575.

Willinger, W., and Taqqu, M.S. (1991). Toward a convergence theory for continuous stochastic securities martet models, *Math. Finance* **1** no. 1, p. 55–99.

Department of Statistics
University of Chicago
5734 University Avenue
Chicago, Illinois 60637